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How-To-Do-It

The Analysis of Leaf Shape Using Fractal Geometry

Gregg Hartvigsen

We often begin studying biological systems, such as molecules, organisms, or even aggregations of organisms in groups, by trying to describe their structure. Structure, or more simply, shape, is often used to describe differences between species. Shape also strongly influences function (e.g. the shape of a male moth's antennae greatly influences his ability to detect the pheromones of females that may be miles away [see Vogel 1988]). The shapes of objects and organisms traditionally have been described using Euclidean geometry. This type of geometry is the basis of what we are all familiar with from high school, and leads to simple shapes like lines, squares, circles and cubes. These structures also define our traditional sense of dimensions in space (e.g. a line is one-dimensional, a square is two-dimensional, etc.). Organisms, however, rarely fit these simple shapes and, instead, are a very complex combination of these forms or usually something altogether different. Try, for example, to think of the shape of a human as made up of spheres, right prisms and cylinders. The person is likely to look silly.

Size also is an important structural component of objects. For organisms, size, by definition, changes during growth and development. Shape, on the other hand, may or may not change. Imagine, for example, what an adult human would look like if she were to retain the same size relationship between head and body from birth to adulthood. Things that do not change shape, or relative sizes of parts as they grow, are referred to as being isometric. When shape does change as a function of size we refer to these objects as having an allometric scaling relationship. We would like to know how to describe the shapes of organisms and how shape changes.

The list of Euclidean shapes does not get us very far, even when we try to describe something simple like a leaf (Figure 1). You could describe this leaf as being similar to an oval or square, but that is clearly inadequate if we are interested in details about the leaf. You could attempt a complicated description using many triangles, for instance, but that would be both difficult and of dubious value. With leaves we may think about the amount of area the leaf covers, but this takes the interesting shape of the leaf and converts it into a Euclidean shape. The leaf in Figure 1 might have an area of 9 cm^2, but this implies the leaf has dimensions of 3 x 3 cm, which is a square. If I have another 9 cm^2 leaf, does it look the same or could it be

Figure 1. Norway maple leaf (*Acer platanoides*). Determining the shape of a simple maple leaf using Euclidean geometry is difficult, although we can easily tell this differs from an oak leaf.

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completely different looking? The latter leaf might be from a different species and be full of holes due to an attack by a herbivore. If shape influences function then simplifying a leaf to its area, measured perhaps in square units, is likely to overlook something very important. This exercise describes a relatively simple, quantitative approach to measure the shape of complicated but real structures. How shape relates to function is an excellent question for further discussion but is beyond the scope of this article.

Here is another example of the problems we have when we try to measure the shapes of things. Think about measuring the length of the East Coast of the United States (see Mandelbrot 1977). If you measure the linear distance from the eastern tip of Maine down to southern Florida, you will find a distance of about 1200 miles. What we have done is to take a 1200-mile-long straight ruler, lay it down, and determine that the coast is about 1200 miles long. Now, if you tried to hike along the coast you would find that you would have to walk quite a bit further. Why? You can't walk in a straight line! If you used a 20-meter ruler to measure the twists and turns of the East Coast, you would estimate the coast to be about 5,000 miles, not 1200 miles. But your 20-meter ruler is still pretty unwieldy. If you used a 1-meter ruler (one stride) you may find the East Coast is on the order of 25,000 miles long. That's about equal to the circumference of the Earth (assuming the Earth is a sphere, which, by the way, it is not). So, what is the actual length of the East Coast? Well, it turns out that there isn't one answer because the answer is scale-dependent. The length depends on the length of your ruler.

We call irregular shapes, such as a coast line, fractals. Students from elementary school through college should be able to think of such examples because most things in nature are fractal, and they are not easily described using the simple shapes of Euclidean geometry.

To understand fractal geometry, however, we have to first review some Euclidean geometry. In Euclidean geometry we talk about dimensions. There are four that should be familiar. The first three are a line (one dimension: length), a flat surface such as a square (two dimensions: length and width), and solids such as cubes (three dimensions: length, width and height). Students are usually familiar with time as the fourth dimension.

In this traditional sense trees are generally three-dimensional objects while leaves are two-dimensional. Let's simplify dimensions by using the letter D. To determine length (1-D), area (2-D), or volume (3-D), we can use a simple formula. Let us start with a line two units long (Figure 2a), and we will define s as the length of a unit on our ruler so that total length $= s^D$. If $s = 2$ and $D = 1$, then length $= s^D = 2^1 = 2$.

Two-dimensional objects such as squares have $D = 2$ (Figure 2b). If we again let s = length of a unit on a side, then we can find the area of a square with our formula such that area $= s^D = 2^2 = 4$. We can make a cube the same way (Figure 2c). Then volume $= s^D = 2^3 = 8$. This gentle introduction, as I have found, has been well received by both freshmen and seniors at the college level and is fundamental in understanding the difference between Euclidean geometry and fractal geometry.

In the above examples we calculated the length of a 1-D line, the area of a 2-D surface, and the volume of a 3-D solid. The objects, however, all had straight-line sides. But organisms rarely have such simple forms. We could try to find the *area* of an irregularly shaped leaf, for example, by counting the number of 2-D squares that the leaf covers on a piece of graph paper, but the answer we would get would depend on the sizes of the squares on the graph paper, just like the answer for the length of the East Coast depends on the size of a ruler. The area of leaves is important to individual plants but says nothing about the actual shape of the leaf, which is more likely to influence function, such as the movement of materials into and out of leaves.

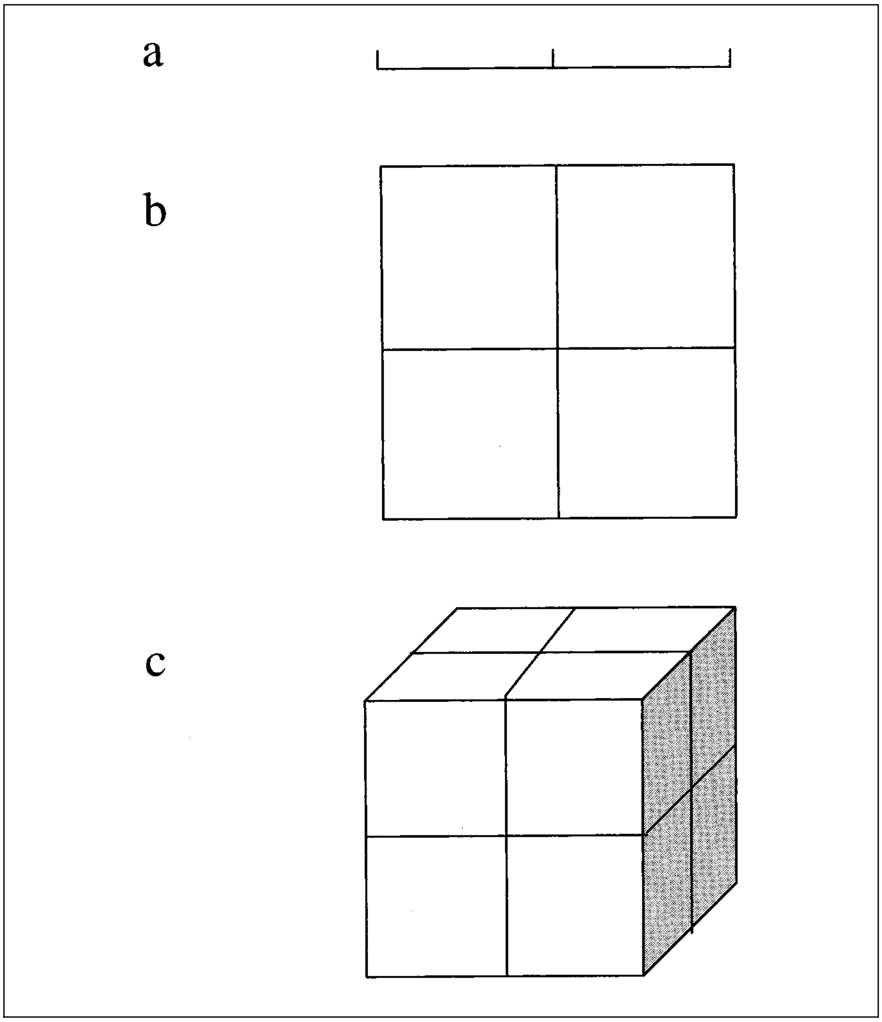

Figure 2. Euclidean geometric shapes with $s = 2$. The measurements of length (a), area (b), and volume (c) of these shapes can be determined using the equation s^D where s is the length of one side unit and D is the standard, Euclidean dimension.

We now can turn to fractal geometry to help understand the simple question, what is the shape of leaves? We use the same D as before for dimension, but now let D vary continuously rather than restrict it to integers or whole numbers. If we take a straight line, for example, we can measure its length using standard Euclidean geometry, which tells us both its shape and length. But if we begin to bend the line just a little bit we immediately lose our ability to describe the line's shape. Fractal geometry enables us to assign a quantitative measure of our line's new shape. The line now may have the same length as before but its dimension (fractal D) increases, say to 1.1. If the line is really curvy, it will begin to fill flat space, becoming more like a surface. If it kept twisting and turning, eventually it would fill a surface completely and have a $D = 2$. Now, if the line that filled a surface began to move outward from the surface toward you, making what we recognize as the beginning of a 3-D structure, the fractal D will increase beyond 2 toward 3. The continuous nature of fractal dimensions, therefore, reveals interesting information about the shape of objects. Determining the shape of objects like leaves turns out to be quite simple and relies on a few very useful and powerful quantitative tools. I use this exercise also to introduce the scientific method to my students.

Table 1. Sample data for the hands depicted in Figure 3a and 3b. The heading values are s, the length of the side of one box on the graph paper (e.g. 1 for one inch), and 1n stands for natural logarithm. Fractal dimension D is the slope of the line of the 1n (number of boxes, either closed or open hands) on the y-axis versus 1n (1/s) on the x-axis. For the closed hand data $D = 1.65$ for the open hand data. The interpretation is that the closed hand more fully fills space than the open hand.

s	In(1/s)	Number of Boxes (closed hand)	In (number of boxes, closed hand)	Number of boxes (open hand)	In (number of boxes, open hand)
1.0	0.00	34	1.53	46	1.66
0.5	0.30	116	2.06	146	2.16
0.25	0.60	383	2.58	445	2.65

Method for Determining Fractal Dimension

A. A Fun Assignment for Students To Complete at Home

I ask my students to do this simple exercise before the laboratory exercise. I ask the students to measure the fractal dimensions of their hands held in two different positions. Students must be supplied with three scaled pieces of graph paper where sides of the little boxes are 0.25, 0.5 and 1.0 inches (the use of English or Metric Units is of no concern because fractals are scale independent). Students should set up a data table with six columns following Table 1. The value s equals the length of a side of one small box (or square) on the different pieces of graph paper and ln stands for the natural logarithm (base e). Use a calculator or computer (e.g. using Excel) to obtain logarithms.

Students measure the number of boxes when the hand is closed and when it is open (Figure 3a & 3b) on each of the three types of graph paper. Fractal dimension D is calculated using the box-count method (see Peitgen et al. 1992; Hastings & Sugihara 1993). Each student should place his/her hand anywhere on each piece of graph paper and *lightly* trace the outline of that hand with a pencil. Next students must count all the boxes inside the traced area and count all the boxes that have any part of a line in them, as well. The number of boxes counted should be recorded in the column ''number of boxes: closed hand'' for each size of graph paper (length ''s''). After counting boxes, the faint lines of the closed hands should be erased and the same method applied to open hands, recording these data in the column ''number of boxes: open hand.''

The shape of a closed hand is different from the shape of an open hand, although the area is not different. The fractal dimension (D) will show the difference, if calculated correctly. Fractal D is estimated using a graphical method. Using the ''closed hand'' data, graph ''ln (number of boxes: closed hand)'' on the y-axis and ''ln(1/s)'' on the x-axis. There should be three points that form two lines. The fractal dimension (D) is simply the slope of the best-fit line through the three points (the three points should fall approximately on the same line). To estimate this slope students should calculate the average of the slopes for the two separate lines (students should be able to calculate the slope by hand by dividing the change in y by the change in x). Follow the same method for data collected from the open hand measurements. The fractal D estimates should be different, with the closed hand having a higher value than data from the open hand.

I like to ask my students if, after seeing the differences in D for open and closed hands, they are convinced that their hands really changed shape. This might lead into a discussion on replication. Critically thinking students should realize the flaw of a sample size of one and realize that they cannot say anything about hands in general from simply measuring their own hands. This might lead to an important activity where students combine data from their group or the entire class. Employing a relatively simple *t*-test would be helpful to resolve whether open and closed hands have *statistically* different shapes.

B. In the Class or Laboratory

The lab should begin with the discussion about the change in shape of their hands when open and closed. Leaves also are different and can be quite variable, even within plants. This is clearly seen (Figure 4) in the biennial wild carrot or Queen Anne's lace (*Daucus carota*), a common weedy species that has an extensive range throughout North America and Eurasia.

Students test the null hypothesis that leaf shape, as measured by fractal dimension D, does not change along the vertical stem of *D. carota*. I help my students develop this hypothesis, and then encourage them to put forth alternative hypotheses. I collected 10 plants and mounted the leaves in ascending order on herbarium paper (Figure 4). This could easily be part of the lab, although having the leaves pressed and on hand allows the measurements to be made within an hour, depending on the number of leaves you choose to collect from each plant. I have used just the top and bottom leaves and found the relationship to be strong and rewarding for the students. For upper division college students, I like to have them do multiple leaves and look at the relationship of fractal D along the stem of plants rather than just the top and bottom of the plant.

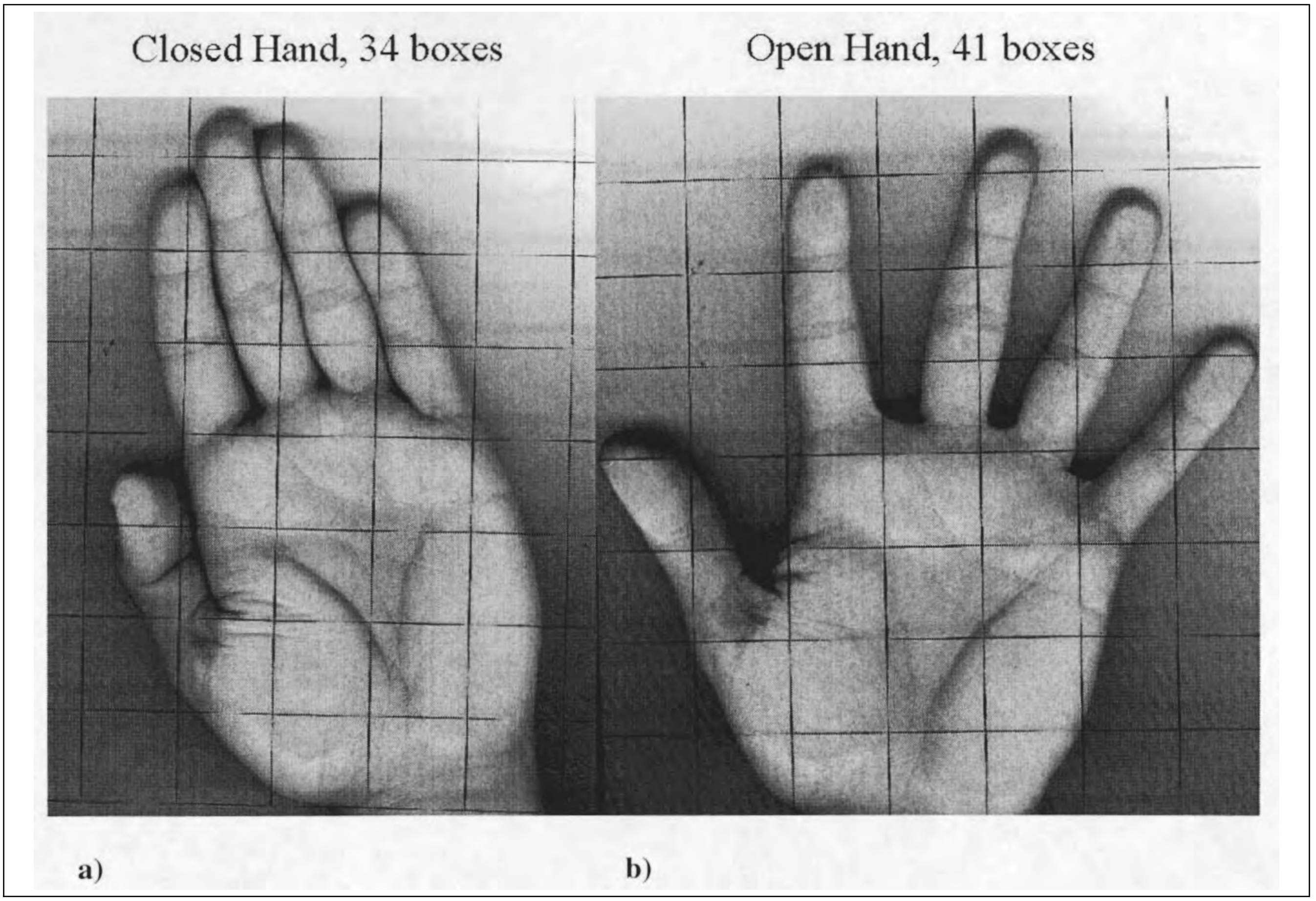

Figure 3. The author's hand in closed (a) and open (b) positions.

To test the null hypothesis I provide the same graph paper reproduced on clear overheads so that the graph paper can overlay the leaf specimens for counting. I also use an additional smaller scale (0.125 inches) for increased resolution of D on the more divided leaves. The data can be analyzed rapidly in a spreadsheet such as Excel. Having students prepare data sheets before lab is helpful and similar to the data sheet used for the hand shapes. The data sheets should have columns for "plant #," "leaf #" (from (the top of the plant), "s" and "# boxes." The leaf at the top of the plant should be labeled #1 (Figure 4). When the data are entered into the spreadsheet it will be easy to create the columns for 1/s and the logarithms [e.g. "=ln(1/A2)" for the ln(1/s)].

Counting Boxes

Students will need to count boxes. This time you should have them use four graph paper sizes (i.e. sides of length 1.0, 0.5, 0.25 and 0.125 inches). Randomly place the graph paper labeled "s = 1.0" over the leaf being measured and count the number of boxes that the leaf occupies. Be sure to count boxes that contain even the smallest amount of leaf edge. Do this twice, again placing the graph paper randomly on each specimen. For each of the four graph papers, average the two counts. Record the average in your data table for each leaf. Repeat the above for each leaf, recording the leaf and plant number.

There are two approaches for analyzing these data. Either way, however, students should be required to generate a graph of the ln(number of boxes counted) vs. ln(1/s) for each leaf (four data points should be generated per leaf for each size of graph paper used). For ln(1/s) you should use the natural logarithm and be careful to take the inverse before logging the value. The slope of this line equals the fractal D for that leaf. You may have students either average the three individual slopes, similar to the technique suggested for estimating the slope with the hand data, or you may have the students enter the data into a calculator or computer and use a standard spreadsheet application.

You may want to ask students whether the points lie on a straight line. If they do not, you may generate some discussion as to why not (variability due to sampling error or there may exist a real change across the scale measured). You also may want to discuss the relationship (either linear or non-linear) between fractal D and leaf position (Figure 5).

A final statistical test may be performed using an analysis of variance (ANOVA), such as is shown in Figure 5. The mean fractal dimension (D) is shown along the vertical stem with 95% confidence error bars representing variability about the means. The inference that can be drawn from this analysis is that the mean fractal D, or shape of the leaves, changes along the vertical stem of *D. carota*.

The Meaning of Fractals in Biology

Our understanding of the shape of structures, organisms and even communities in biology is surprisingly limited. This is quite amazing considering, for example, that we generally consider an organism's structural anatomy well before we investigate function.

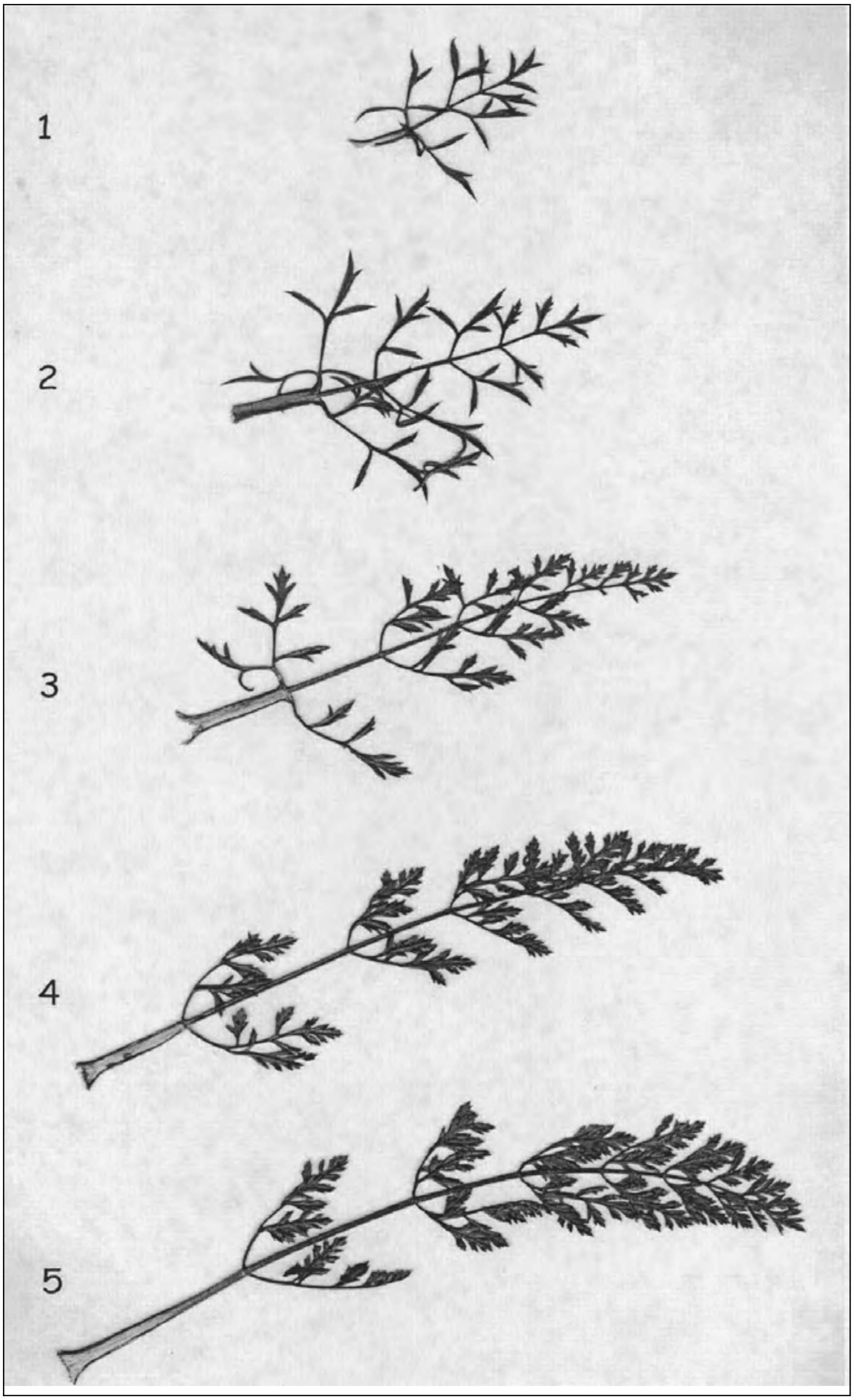

Figure 4. Scanned image of an herbarium sheet with five pressed leaves of Queen Anne's Lace, *Daucus carota*. Leaf number 1 was the highest leaf on the plant. Plants were collected in late August on the Leary farm in Geneseo, New York.

We are hard pressed to answer a student's question as to what is the shape of organs such as mammalian lungs or the structure of a forest. Fractal geometry provides a relatively simple approach to at least describing such shapes. Our challenge will be to, along with our students, deduce what this new-found estimate of shape—the fractal dimension—tells us about function.

It is important to note that this exercise outlines an approach to estimate the shape of objects and not their function. Fractals, in a strict mathematical sense, are defined as objects having self-similar properties at all scales. This self-similarity property, however, clearly does not hold for real biological structures and breaks down long before the level of cells. The fact that organisms are fractal at certain scales, however, may imply the presence of both important developmental constraints and the presence of conservative evolutionary processes governing the shape of organisms. Examples of self-similar structures include a small leaflet on a fern frond that closely resembles a miniature version of the entire frond or the branch of a tree that resembles the entire tree. It also is possible to consider investigating the change in shape of organisms or structures, using fractal geometry, to investigate individual response to changing environmental factors. Our current limited understanding of the relationship between structure, measured using fractal geometry, and function should not, I believe, hinder our appreciation of the importance of estimating the shape of biological structures. We first must identify what we have (the structure) and then, with our students, tackle what it means for the organism.

Conclusion

I believe this exercise is useful and practical for a broad range of students. Students at all levels can enjoy counting the number of boxes on graph paper intersected by leaves. The determination of fractal dimension may be done either by calculating the slope of lines by hand (change in y over change in x and compared for just leaves from the tops and bottoms of plants) or with a statistical program on the computer. There also are opportunities for higher-level students to test hypotheses using leaves of replicated plants to look for non-linear trends in leaf shape along the stem of plants, across species, and under different environmental growth conditions. The importance of shape also makes for interesting discussions because shape discrimination is so easy for us to do visually, yet has generally escaped quantitative analysis in science. Finally, in this exercise students will most likely find that shape does in fact change along the stem of this plant which can lead them to question how shape matters for both plants and the organisms that depend on them.

Acknowledgments

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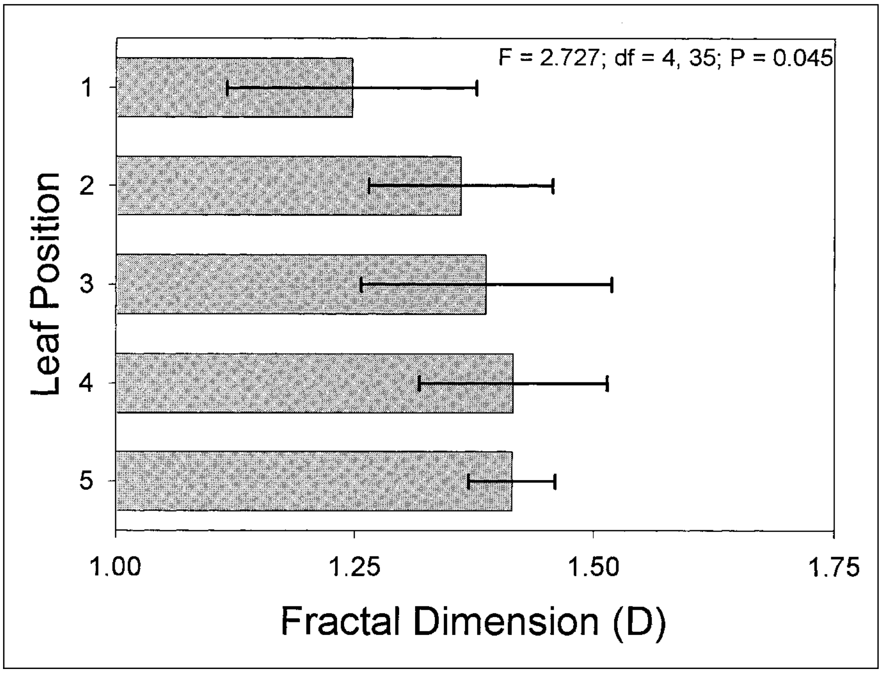

Figure 5. The mean fractal dimension (D) for the top five leaves determined by students for eight plants. Error bars represent 95% confidence intervals.

manifestation of this laboratory and to two anonymous reviewers whose comments improved the manuscript considerably. Many thanks to the Leary's and their many acres of Queen Anne's lace. I also would like to thank the students of my spring 1999 Population and Community Ecology course at SUNY Geneseo for their work on this laboratory and providing the data presented in Figure 5.

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