Spherical Radiation as a Model for Gravitational Waves

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ABSTRACT

Our ultimate goal is to model the dynamics of a compact object orbiting a large black hole. However, there is no known exact solution to the Einstein field equation and it requires application of numerous mathematical techniques. Therefore, in this project, we seek to solve the simple differential equation called wave equation in spherical coordinate. This model is analogous to Einstein's field equation for a black hole binary system and by obtaining a three dimensional wave equation, we take a step toward a theoretical model of gravitational waves from astronomical sources

INTRODUCTION

Wave equation in Cartesian Coordinate

A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity

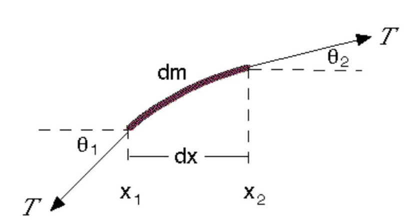
Derivation of wave equation in Cartesian coordinate

$$F = ma$$

$$\mu\, \partial x \frac{\partial f^2}{\partial t^2} = T\sin\theta_2 - T\sin\theta_1$$

$$= T\left(\frac{\partial f}{\partial x}|_{x+\partial x} - \frac{\partial f}{\partial x}|_x\right)$$

$$= T\frac{\partial f^2}{\partial x^2}\,\partial x$$



$$\frac{\partial f^2}{\partial x^2} = \frac{1}{v^2}\frac{\partial f^2}{\partial t^2}$$

3 D wave equation is analogous to 1D wave equation and can be obtained by adding two more spatial terms

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

Wave equation in spherical coordinate

Waves in the 3-D space propagating from a point source at the origin.

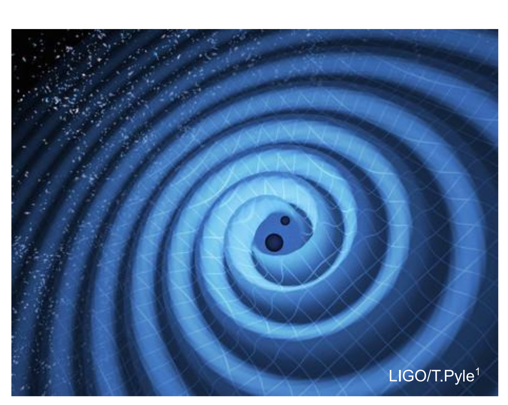
Analogous to the gravitational wave, which propagates from two black holes orbiting each other.

Gravitational Wave

Disturbances in the curvature of spacetime that propagates outward from their sources at the speed of light.

The sources: Binary system composed of black holes, neutron stars or white dwarfs

First detected by Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2016

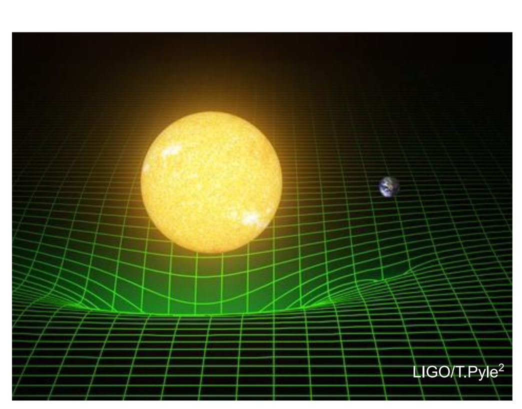


Einstein's Field Equation

When gravitational fields become very strong or speeds become very fast, Einstein's theory of General Relativity describe gravity

Einstein's field equation: Spacetime metric, which involves second-order non-linear partial equation

There is no known exact solution to the Einstein's field equation and it can be solved only numerically

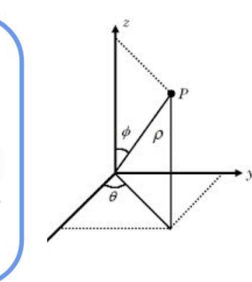


METHOD

Wave Equation in Cartesian Coordinate

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$
$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$
$$\varphi = \tan^{-1}\frac{y}{x}$$



Wave Equation in Spherical Coordinate

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

We assume that $f(t,r,\theta,\varphi)$ can be expressed as product of function with dependence on fewer variables

$$f(t,r,\theta,\phi) = X(t,r)Y(\theta,\varphi)$$

Substitution into the wave equation and multiplying by $r^2/(XY)$

Separation of Variables

$$\frac{r^2}{X}\frac{\partial^2 X}{\partial r^2} + \frac{2r}{X}\frac{\partial X}{\partial r} - \frac{r^2}{X c^2}\frac{\partial^2 X}{\partial t^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial \theta^2} + \frac{1}{Y\tan\theta}\frac{\partial Y}{\partial \theta} + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2} = 0$$

$$X(t,r) = T(t)R(r)$$
$$Y(\theta,\varphi) = P(\theta)\Phi(\varphi)$$

$$0 = \frac{\partial^2 P}{\partial \theta^2} + \frac{1}{\tan\theta}\frac{\partial P}{\partial \theta}\left(C_1 - \frac{C_3}{\sin^2\theta}\right)P$$

$$0 = \frac{\partial^2 \Phi}{\partial \varphi^2} + C_3\Phi$$

$$0 = \frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} + C_2 T$$

$$0 = \frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} + \left(C_2 - \frac{C_1}{r^2}\right)R$$

Spherical Harmonics

$$Y_l^m(\theta,\varphi) = C_l^m P_l^m(\theta)e^{im\varphi}$$

$P_l^m(\theta)$ is associated Legendre polynomials with the argument of $\cos\theta$, and C_l^m is a constant.

R only depends on r for given values of l and ω
Near $r = \infty$, R can be described as follow

Radial Equation

$$R_{l\omega} \approx \left(\frac{c}{\omega r}\right)e^{\pm\frac{i\omega r}{c}}\sum_{j=0}^{j_{max}} b_j^{\pm}\left(\frac{c}{\omega r}\right)^j$$

$$b_j^{\pm} = \mp\frac{i(j+1)(j-l-1)}{2j}\, b_{j-1}^{\pm}$$

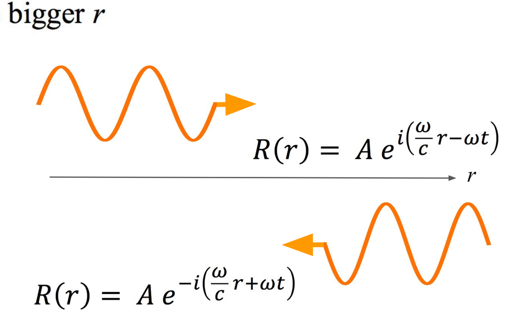
A function can be expressed as a Fourier series if it is a periodic function with period $\frac{2\pi}{\Omega}$ where $\omega_n = n\Omega$

$$g(t) = \sum_{n=-\infty}^{\infty} A_n e^{-i\omega_n t}$$

Fourier series consists of a complete orthonormal set of periodic functions

A_n is analogous to the spherical harmonic coefficient

Though $R_{l\omega}$ includes $b_j^{\pm}$ term, we only keep b_j^{+} term since the waver is propagating from smaller r to bigger r



$$R(r) = A\,e^{i\left(\frac{\omega}{c}r - \omega t\right)}$$

$$R(r) = A\,e^{-i\left(\frac{\omega}{c}r + \omega t\right)}$$

Fourier Series

$$f(t,r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-\infty}^{\infty} C_{lmn}R_{ln}(r)Y_{lm}(\theta,\varphi)e^{-i\omega_n t}$$

Spherical Wave Equation

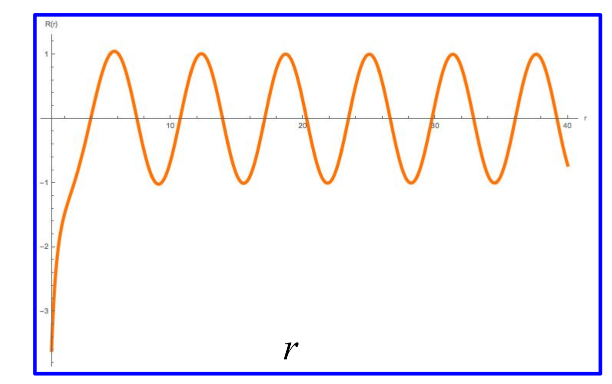
$$f(t,r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=-\infty}^{\infty}\sum_{j=0}^{j_{max}} C_{lmn}\,Y_{lm}(\theta,\varphi)\,b_j^{+}\left(\frac{c}{\omega r}\right)^{j+1} e^{i\left(\frac{\omega}{c}r - \omega t\right)}$$

RESULTS

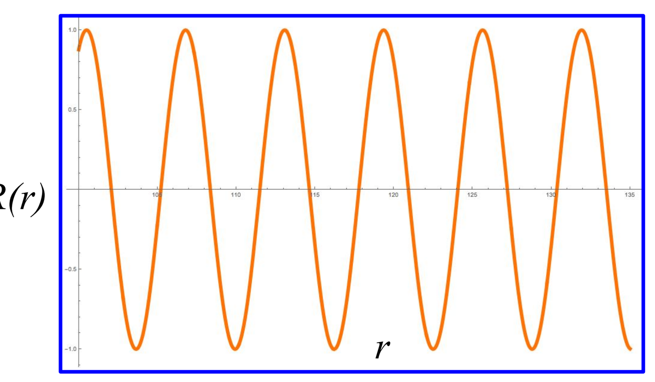
In this project, we assume $l = 2$, $m = 1$, $C_{lmn} = 1$, ω = 1

Dependence of the wave function on the radius

Near the source



Far from the source



Far from a source, $R(r)$ shows smaller amplitude as r increases

$$R(r) = \frac{c}{\omega}\frac{1}{r}\left(1 + \left(\frac{c}{\omega r}\right)^1 b_l + \left(\frac{c}{\omega r}\right)^2 b_2 + \left(\frac{c}{\omega r}\right)^3 b_3 \ldots + \left(\frac{c}{\omega r}\right)^l b_l\right)e^{\frac{i\omega r}{c}}, \quad b_{l+1} = 0$$

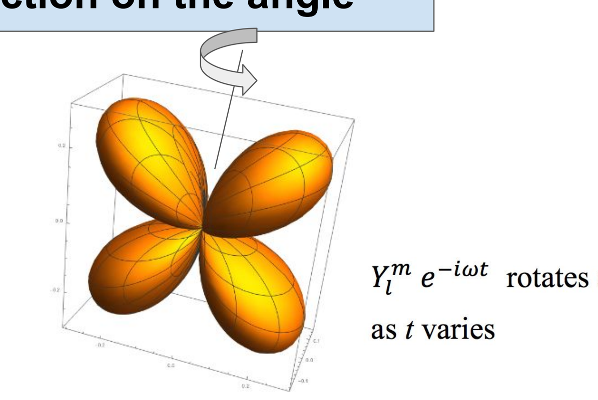
At $r \approx 0$ $r\,R(r)$ This term blows up, explaining non-sinusoidal behavior near r = 0

At $r \approx \infty$ $r\,R(r) = \frac{c}{\omega}e^{\frac{i\omega r}{c}}$ Sinusoidal function with a constant amplitude

Dependence of the wave function on the angle

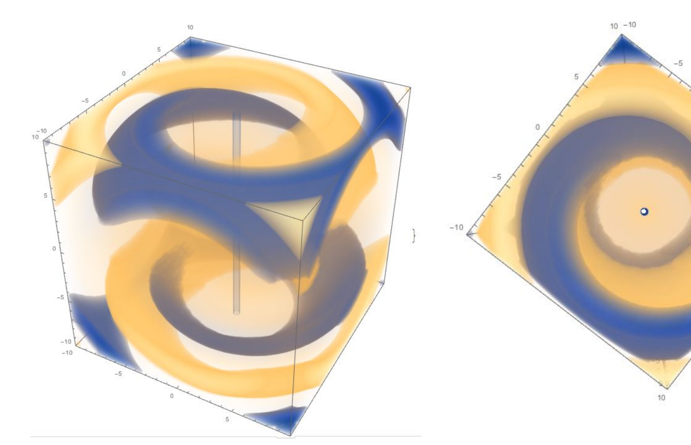
$l = 2, m = 1$

$$Y_l^m = -\frac{1}{2}e^{i\varphi}\sqrt{\frac{15}{2\pi}}\cos\theta\sin\theta$$



$Y_l^m e^{-i\omega t}$ rotates around z axis as t varies

Spherical Wave



Blue: Positive wave function
Orange: Negative wave function

The image: Spherical waves multiplied by r^2 so that the waves don't become too small to be visualized as r increases

Animation https://bit.ly/3ej55yR

CONCLUSION

We have successfully solved the wave equation in spherical coordinate

Future works include utilizing these results to obtain the theoretical model of gravitational wave

REFERENCES

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