Calculating Gravitational Waves in a Black Hole Binary System

Nami Nishimura and Thomas Osburn

Geneseo college of State University of New York Department of Physics and Astronomy

ABSTRACT

Our ultimate goal is to compute the gravitational waves in an extreme mass-ratio binary system. Since gravitational perturbations are difficult to calculate, we model the compact object as a point carrying a scalar charge q and moving around a spinning black hole.
In this project, we take advantage of symmetry under rotation around the spin axis in order to separate ϕ variable and leverage the periodicity of the source to separate t variable with a Fourier series. The remaining differential equations will be solved numerically with an appropriate discretization considering a grid of points in r-θ plane

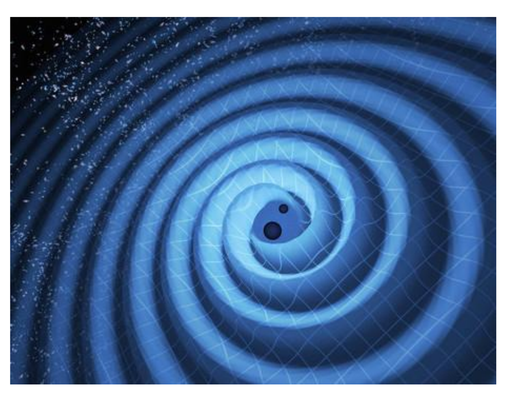
INTRODUCTION

Gravitational Wave

Disturbances in the curvature of spacetime that propagates outward from their sources at the speed of light.

The sources: Binary system composed of black holes, neutron stars or white dwarfs

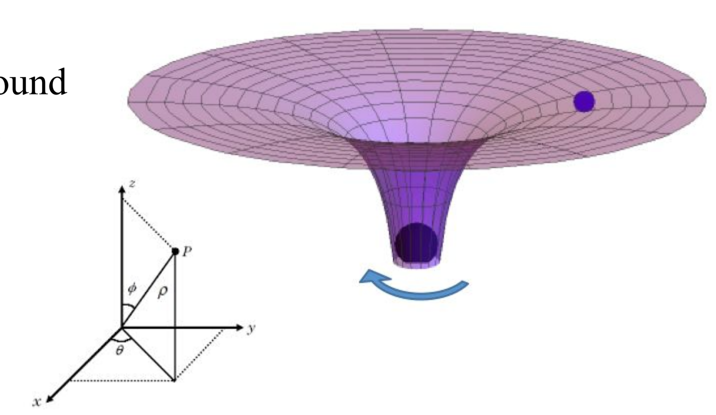
First detected by Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2016



Our Theoretical Model

A point particle with a scalar charge q moving around a massive spinning black hole in a binary system.
It shows azimuthal symmetry.
Point particle is located at $r=r_0$ and $\theta=\pi/2$

M: Mass of the spinning black hole
a: Spin Parameter



METHOD

The Scalar-Field Equation

In Boyer-Lindquist coordinate[1], the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a Kerr spacetime is given by

$$ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4aMr\sin^2\theta}{\rho^2}\,dt\,d\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2\,d\theta^2 + \left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,d\phi^2 \quad \text{....eq(1).}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$

1. Generalization of the coordinates used for the metric of a Kerr black hole

The scalar field is governed by the sourced Klein- Gordon equation[2]

$$\Box\Phi = \frac{1}{\sqrt{-g}}\delta\left(\sqrt{-g}\,g^{\mu\gamma}\delta_\gamma\Phi\right) = S \quad \text{....eq(2).}$$

where $g = -\rho^4\sin^2\theta$ is the Kerr metric determinant, and $g^{\mu\nu}$ is the covariant form of the metric.
Solving eq.(1) and eq.(2), below is the obtained m-mode scalar wave operator $\Box_\Psi^m$

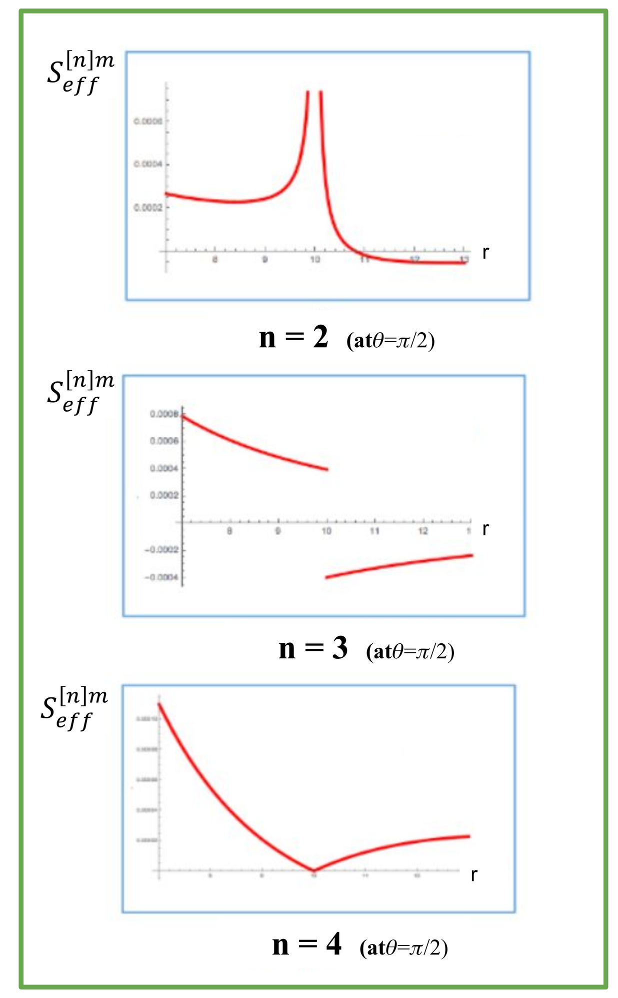
$$\Box_\Psi^m = \frac{\partial^2}{\partial t^2} + \frac{4iamMr}{\Sigma^2}\frac{\partial}{\partial t} - \frac{(r^2+a^2)}{\Sigma^2}\frac{\partial^2}{\partial r_*^2} - \left[\frac{2iamr(r^2+a^2)-2a^2\Delta}{r\Sigma^2}\right]\frac{\partial}{\partial r_*} - \frac{\Delta}{\Sigma^2}\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} - \frac{m^2}{\sin^2\theta} - \frac{2M}{r}\left(1-\frac{a^2}{Mr}\right) - \frac{2iam}{r}\right] \quad \text{....eq(3).}$$

2. Relativistic wave equations.

Effective Source

We need to obtain a local expansion for Source-field in the vicinity of a point on the worldline. We expand them up until fourth order to gain finite Effective Source by using Taylor expansion.

Right plots show that for n=4, the Effective Source is finite



m-Mode Decomposition

We make the standard coordinate change $\phi \rightarrow \varphi$ to take into consideration that Boyer-Lindquist azimuthal coordinate ϕ is pathological at the event horizon. Since both scalar field $\mathbf{\Phi}$ and the source S are periodic in φ, they admit m-mode decomposition. Using Fourier inverse formula, eq(3) can be expressed as

$$S_{eff}^{[4]m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Box\Phi_p^{[4]} e^{-im\varphi}\,d\phi$$

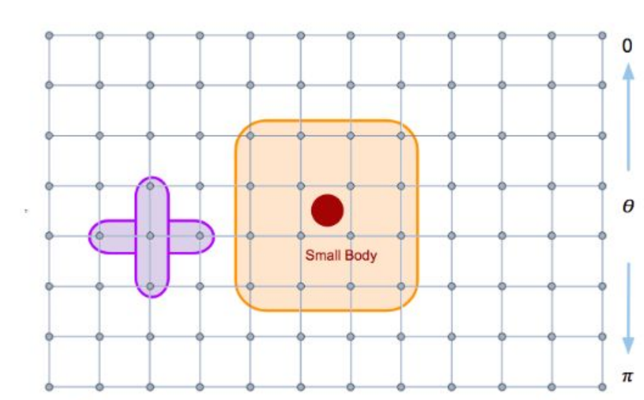
$$d\varphi = d\phi + \frac{a}{\Delta}dr$$
$$\varphi(\phi, r) = \phi + \frac{a}{r_+ - r_-}\ln\left|\frac{r-r_+}{r-r_-}\right|$$

Finite Difference Scheme

The scalar field at the grid point corresponding to the coordinate value r* and θ is computed based on the neighboring scalar potentials, obtained in previous steps

All spatial derivatives in eq(3) are replaced with finite-difference approximation of second-order accuracy

$$\Psi_{r*}^m = \frac{\Psi_{r^*+\Delta r^*,\theta} - \Psi_{r^*-\Delta r^*,\theta}}{2\Delta r^*}$$
$$\Psi_{r^*r^*}^m = \frac{\Psi_{r^*+\Delta r^*,\theta} - 2\Psi_{r^*,\theta} + \Psi_{r^*-\Delta r^*,\theta}}{\Delta r^{*2}}$$
$$\Psi_\theta^m = \frac{\Psi_{r^*,\theta+\Delta\theta} - \Psi_{r^*,\theta-\Delta\theta}}{2\Delta\theta}$$
$$\Psi_\theta^m = \frac{\Psi_{r^*,\theta+\Delta\theta} - 2\Psi_{r^*,\theta} + \Psi_{r^*,\theta-\Delta\theta}}{\Delta\theta^2}$$



At $r^* = \pm\infty$, we assume that the waves are only propagating radially outward from the source and we describe wave equations as

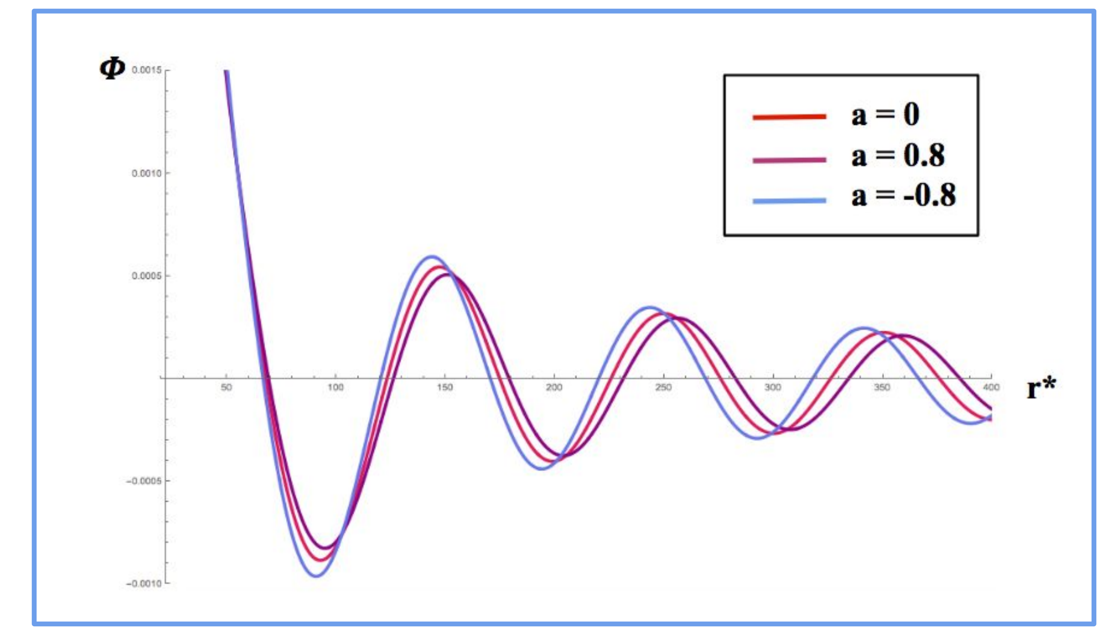
$$\Psi^\pm = f(\theta)e^{-i(\omega t \mp \omega r^*)}$$

Thus, boundary conditions are

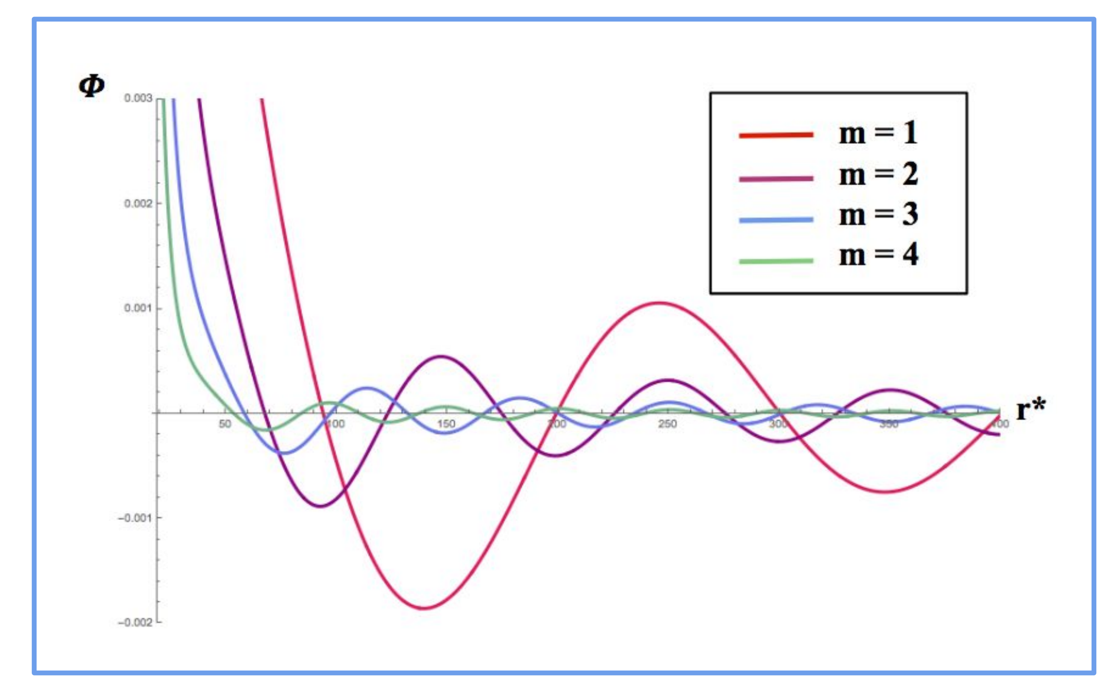
$$\Psi^+|_{r^*=\infty} = \left(-\frac{i}{\omega}\right)\frac{\partial\Psi^+}{\partial r^*} \qquad \Psi^-|_{r^*=-\infty} = \left(\frac{i}{\omega}\right)\frac{\partial\Psi^-}{\partial r^*}$$

r* is so called tortoise coordinate that maps the interval (r_+, ∞) to the interval $(-\infty,\infty)$, where r_+ is the event horizon

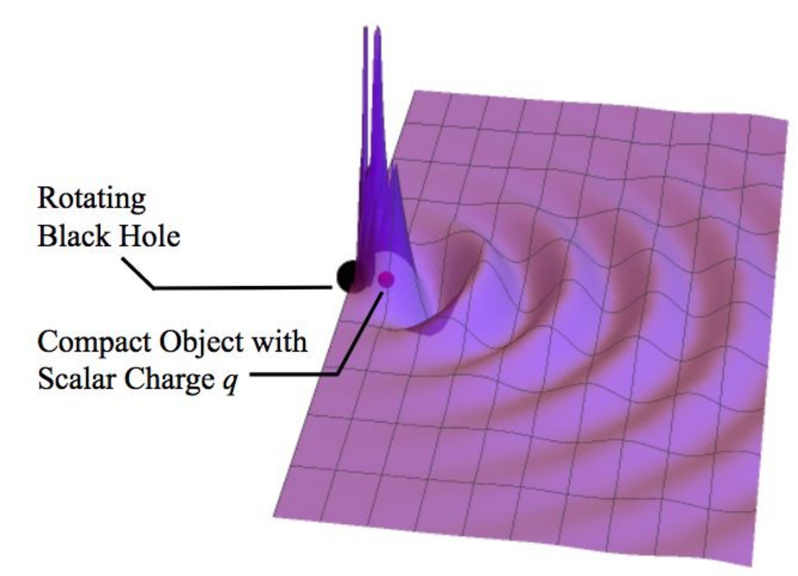
RESULTS



Scalar Field $\mathbf{\Phi}$ as a function or r* with different spin parameter a



Scalar Field $\mathbf{\Phi}$ as a function or r* with different m



Animation

https://tinyurl.com/57f6enue

CONCLUSION

We successfully calculated wave propagations near a rotating black hole in frequency domain by separating t and ϕ variables. Our next goal would be taking the gradient of the obtained scalar field in order to gain scalar-field radiation-reaction force, also known as the self-force

REFERENCES

1.T. Pyle and LIGO, *Spiral Dance of Black Holes* (2016)

2.Sam R. Dolan and Leor Barack, "Self force via m-mode regularization and 2+1D evolution: II Scalar-field implementation on Kerr spacetime", Phys.Rev. D84:084001 (2011) https://arxiv.org/abs/1107.0012

3.LIGO Scientific Collaboration, Physical Review Letters **116**, p. 061102 (2016)