Calculating Gravitational Waves in a Black Hole Binary System

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ABSTRACT

Our ultimate goal is to compute the gravitational waves in an extreme mass-ratio binary system. Since gravitational perturbations are difficult to calculate, we model the compact object as a point carrying a scalar charge q and moving around a spinning black hole.
In this project, we take advantage of symmetry under rotation around the spin axis in order to separate ϕ variable and leverage the periodicity of the source to separate t variable with a Fourier series. The remaining differential equations will be solved numerically with an appropriate discretization considering a grid of points in r-θ plane

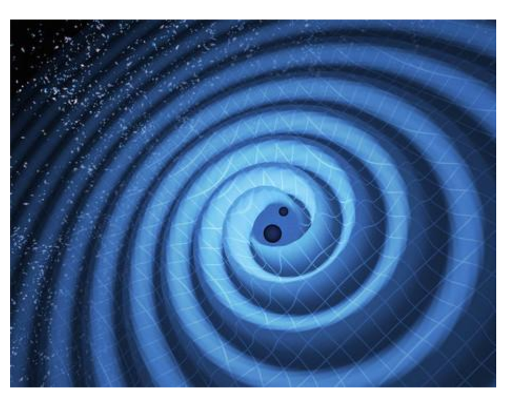
INTRODUCTION

Gravitational Wave

Disturbances in the curvature of spacetime that propagates outward from their sources at the speed of light.

The sources: Binary system composed of black holes, neutron stars or white dwarfs

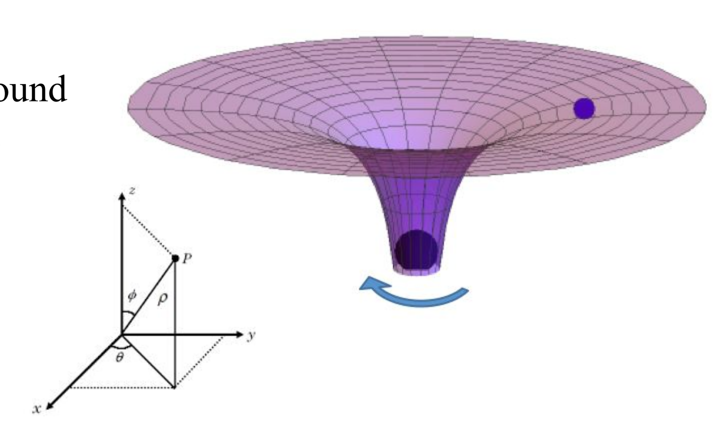
First detected by Laser Interferometer Gravitational-Wave Observatory (LIGO) in 2016



Our Theoretical Model

A point particle with a scalar charge q moving around a massive spinning black hole in a binary system.
It shows azimuthal symmetry.
Point particle is located at $r=r_0$ and $\theta=\pi/2$

M: Mass of the spinning black hole
a: Spin Parameter



METHOD

The Scalar-Field Equation

In Boyer-Lindquist coordinate[1], the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ of a Kerr spacetime is given by

$$ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4aMr\sin^2\theta}{\rho^2}\,dt\,d\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2\,d\theta^2 + \left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\,d\phi^2 \quad \text{....eq(1).}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$

1. Generalization of the coordinates used for the metric of a Kerr black hole

The scalar field is governed by the sourced Klein- Gordon equation[2]

$$\Box\Phi = \frac{1}{\sqrt{-g}}\delta\left(\sqrt{-g}\,g^{\mu\gamma}\delta_\gamma\Phi\right) = S \quad \text{....eq(2).}$$

where $g = -\rho^4\sin^2\theta$ is the Kerr metric determinant, and g^{uv} is the covariant form of the metric.
Solving eq.(1) and eq.(2), below is the obtained m-mode scalar wave operator $\Box^m_\Psi$

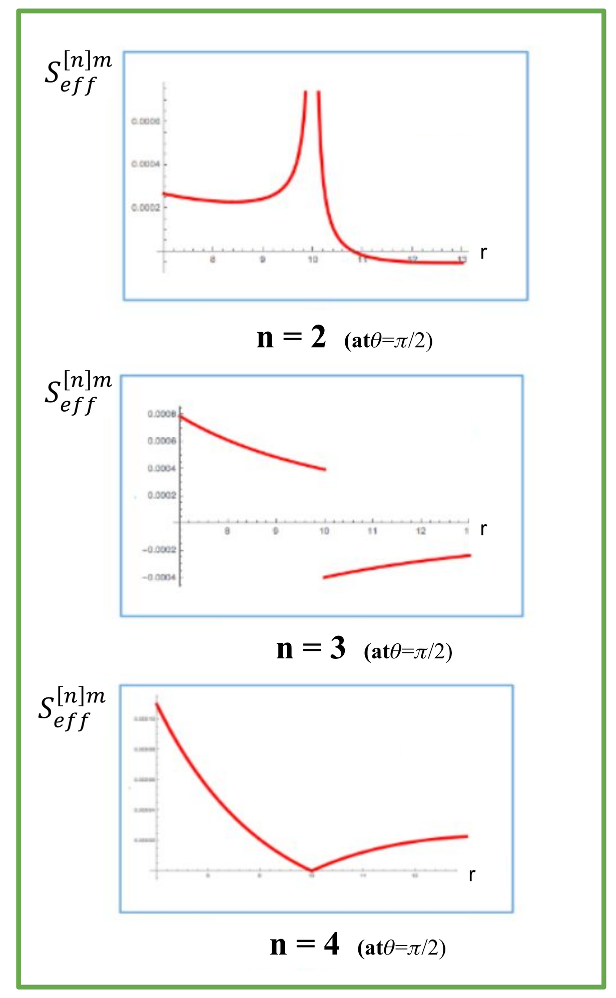
$$\Box^m_\Psi = \frac{\partial^2}{\partial t^2} + \frac{4iamMr}{\Sigma^2}\frac{\partial}{\partial t} - \frac{(r^2+a^2)}{\Sigma^2}\frac{\partial^2}{\partial r_*^2} - \left[\frac{2iamr(r^2+a^2)-2a^2\Delta}{r\Sigma^2}\right]\frac{\partial}{\partial r_*} - \frac{\Delta}{\Sigma^2}\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} - \frac{m^2}{\sin^2\theta} - \frac{2M}{r}\left(1-\frac{a^2}{Mr}\right) - \frac{2iam}{r}\right] \quad \text{....eq(3).}$$

2. Relativistic wave equations.

Effective Source

We need to obtain a local expansion for Source-field in the vicinity of a point on the worldline. We expand them up until fourth order to gain finite Effective Source by using Taylor expansion.

Right plots show that for n=4, the Effective Source is finite



m-Mode Decomposition

We make the standard coordinate change $\phi \to \varphi$ to take into consideration that Boyer-Lindquist azimuthal coordinate ϕ is pathological at the event horizon. Since both scalar field $\mathbf{\Phi}$ and the source S are periodic in φ, they admit m-mode decomposition. Using Fourier inverse formula, eq(3) can be expressed as

$$S^{[4]m}_{eff} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Box\Phi^{[4]}_p e^{-im\varphi}d\phi$$

$$d\varphi = d\phi + \frac{a}{\Delta}dr$$

$$\varphi(\phi, r) = \phi + \frac{a}{r_+ - r_-}\ln\left|\frac{r-r_+}{r-r_-}\right|$$

Finite Difference Scheme

The scalar field at the grid point corresponding to the coordinate value r* and θ is computed based on the neighboring scalar potentials, obtained in previous steps

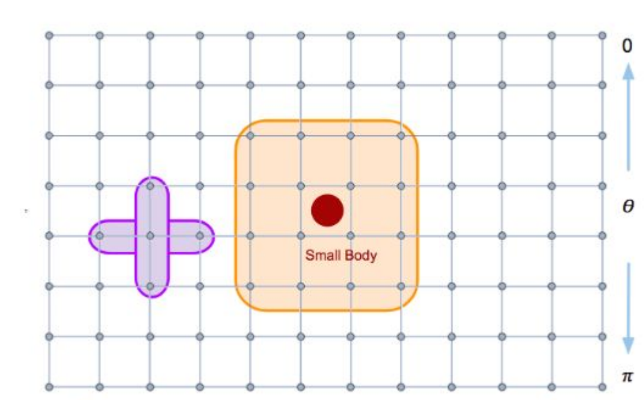
All spatial derivatives in eq(3) are replaced with finite-difference approximation of second-order accuracy

$$\Psi^m_{r*} = \frac{\Psi_{r^*+\Delta r^*,\theta} - \Psi_{r^*-\Delta r^*,\theta}}{2\Delta r^*}$$

$$\Psi^m_{r^*r^*} = \frac{\Psi_{r^*+\Delta r^*,\theta} - 2\Psi_{r^*,\theta} + \Psi_{r^*-\Delta r^*,\theta}}{\Delta r^{*2}}$$

$$\Psi^m_{\theta} = \frac{\Psi_{r^*,\theta+\Delta\theta} - \Psi_{r^*,\theta-\Delta\theta}}{2\Delta\theta}$$

$$\Psi^m_{\theta} = \frac{\Psi_{r^*,\theta+\Delta\theta} - 2\Psi_{r^*,\theta} + \Psi_{r^*,\theta-\Delta\theta}}{\Delta\theta^2}$$



At r* = ± ∞, we assume that the waves are only propagating radially outward from the source and we describe wave equations as

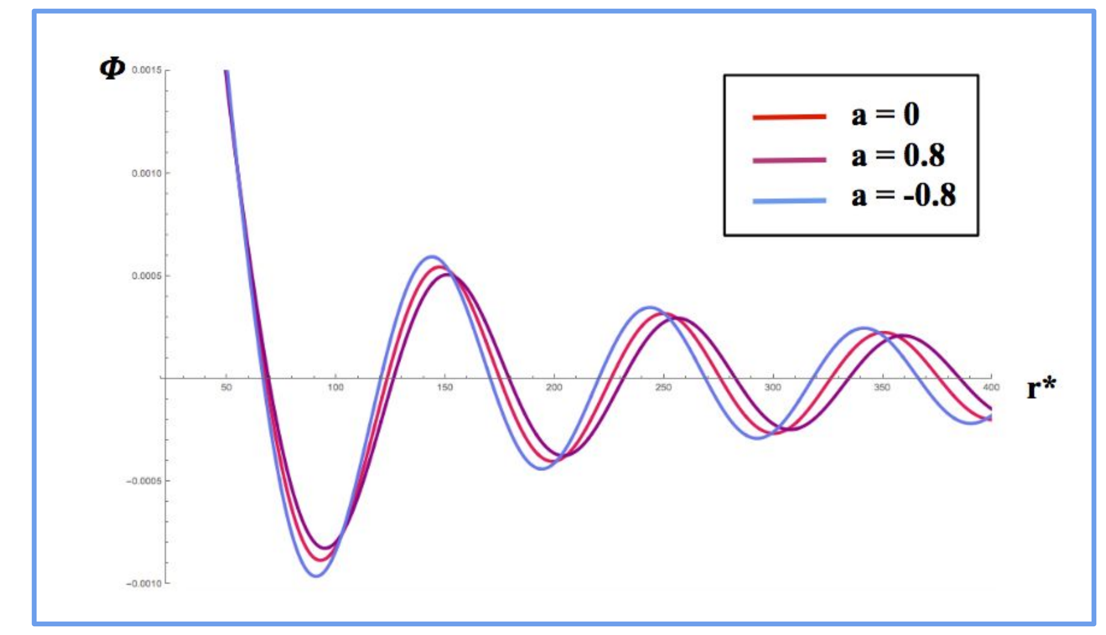
$$\Psi^{\pm} = f(\theta)e^{-i(\omega t \mp \omega r^*)}$$

Thus, boundary conditions are

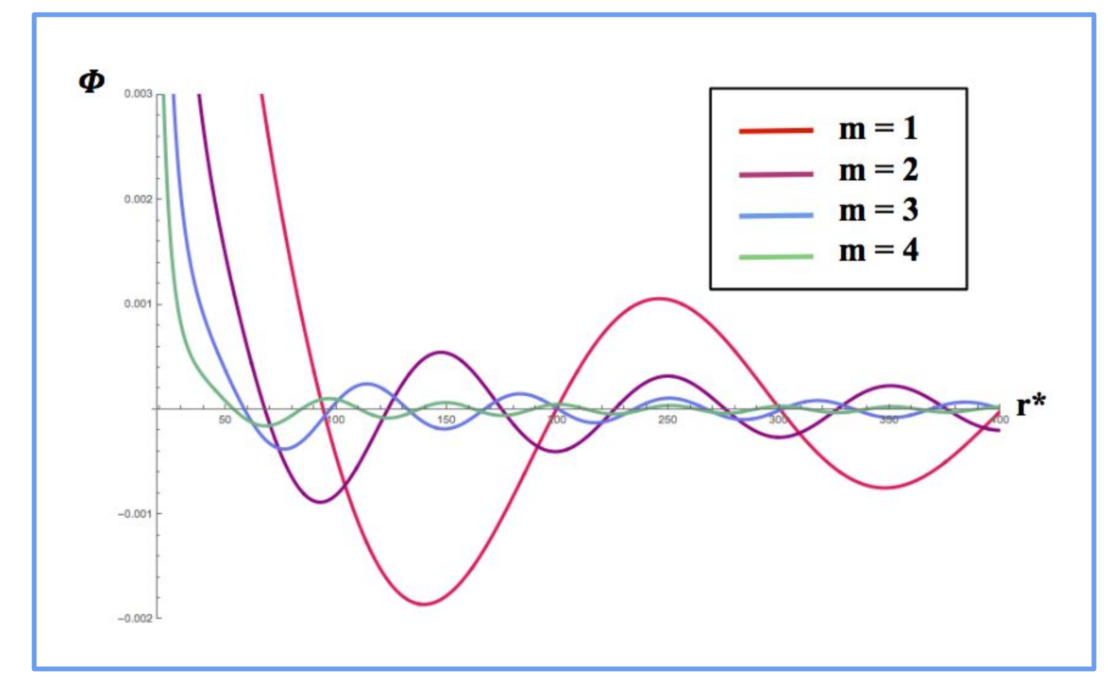
$$\Psi^+|_{r^*=\infty} = \left(-\frac{i}{\omega}\right)\frac{\partial\Psi^+}{\partial r^*} \qquad \Psi^-|_{r^*=-\infty} = \left(\frac{i}{\omega}\right)\frac{\partial\Psi^-}{\partial r^*}$$

r* is so called tortoise coordinate that maps the interval (r_+, ∞) to the interval (-∞,∞), where r_+ is the event horizon

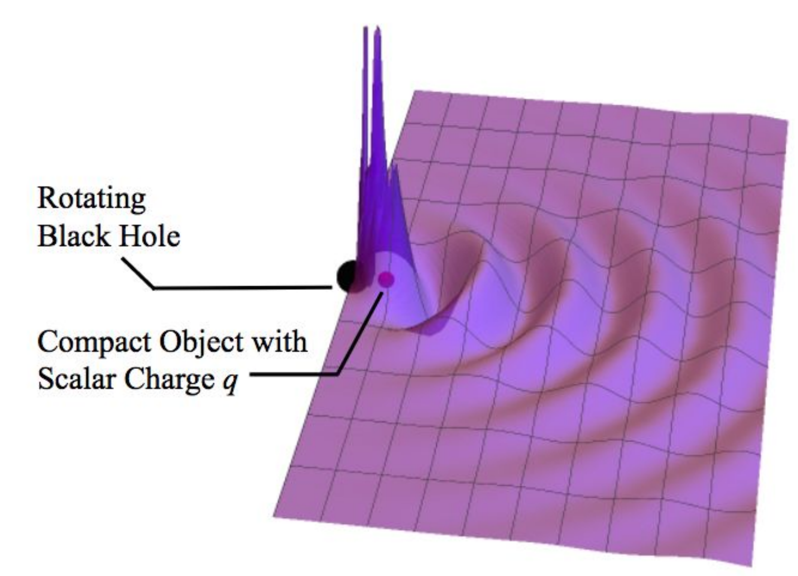
RESULTS



Scalar Field $\mathbf{\Phi}$ as a function or r* with different spin parameter a



Scalar Field $\mathbf{\Phi}$ as a function or r* with different m



Animation

https://tinyurl.com/57f6enue

CONCLUSION

We successfully calculated wave propagations near a rotating black hole in frequency domain by separating t and ϕ variables. Our next goal would be taking the gradient of the obtained scalar field in order to gain scalar-field radiation-reaction force, also known as the self-force

REFERENCES

1.T. Pyle and LIGO, *Spiral Dance of Black Holes* (2016)

2.Sam R. Dolan and Leor Barack, "Self force via m-mode regularization and 2+1D evolution: II Scalar-field implementation on Kerr spacetime", Phys.Rev. D84:084001 (2011) https://arxiv.org/abs/1107.0012

3.LIGO Scientific Collaboration, Physical Review Letters **116**, p. 061102 (2016)