Deflection of Light in the Equatorial Plane of a Kerr Black Hole

Robert Sinesi and Dr. Savi Iyer
Department of Physics & Astronomy, SUNY Geneseo

Abstract

Series expansions for the bending angle of light in the equatorial plane of a Kerr black hole are given for the strong and weak deflection limits with various values of the spin parameter ranging from low to high spin. From the exact bending angle expression, with no known analytical solution, we get series approximations for the bending angle in terms of the impact parameter of the incident light ray. Analytical expressions allow us to derive and connect to other results relating to images formed due to gravitational lensing. Specifically, the asymmetry that arises in the spin-dependent shifts in image positions can be predicted by the analytical expansions. We apply our results for the case of a galactic supermassive black hole to predict asymmetric angular positions of relativistic images on either side of the lens. The possible observation of the asymmetric image shifts with telescopes can only be predicted with the perturbative expansions in the strong deflection regime.

$$\hat{\alpha} = -\pi + 4\sqrt{\frac{r_0}{Q}}\left\{\Omega_1\Big[\Pi(n_1,k) - \Pi(n_1,\psi,k)\Big] + \Omega_2\Big[\Pi(n_2,k) - \Pi(n_2,\psi,k)\Big]\right\} \quad (1)$$

$$\hat{\alpha}(b') = \Big[A_0 + A_1(b') + A_2(b')^2 + \cdots\Big]\log\left(\frac{1}{b'}\right) + \Big[B_0 + B_1(b') + B_2(b')^2 + \cdots\Big] \quad (2)$$

$$\hat{\alpha} = 4\left(\frac{m}{b}\right) + \left[\frac{15\pi}{4} - 4s\hat{a}\right]\left(\frac{m}{b}\right)^2 + \left[\frac{128}{3} - 10\pi s\hat{a} + 4\hat{a}^2\right]\left(\frac{m}{b}\right)^3 + \left[\frac{3465\pi}{64} - 192s\hat{a} + \frac{285\pi}{16}\hat{a}^2 - 4s\hat{a}^3\right]\left(\frac{m}{b}\right)^4 + \mathcal{O}\left[\left(\frac{m}{b}\right)^5\right] + \cdots \quad (3)$$

Introduction

Bending of light around a black hole is extremely important when considering the position of distant objects in the universe. The observed position of the image on the sky is not a good indicator of the location of the source since the path the light takes is influenced by black holes. If the light travels close enough to the black hole, it will travel around the black hole some number of times before exiting, which increases the need for a way to quantify the bending angle. Of course, light that travels near the black hole is of the most interest since the influence of the black hole increases dramatically as the light ray gets closer to the black hole.

The lens equation is a relationship between the actual and apparent position of a distant source on the night sky. If we assume the source is confined in a region that is small compared to the distance involved, the lens equation is valid. Once we obtain an expression for the bending angle of the light, we can quantify the position of the source assuming the other variables of the lens equation have been determined through various other techniques.

Procedure

In order to compute an analytical expression for the bending angle, Mathematica was used. The computations were much too extensive to reasonably do by hand, so the series expansion capabilities were utilized to find the weak and strong deflection limit expressions. For the strong deflection limit, a transformation was done on the impact parameter, b, in order to change the limits from b_{crit} and ∞ to 0 and 1. Since b_{crit} is not easily expanded around, the transformation simplified the computation enough to find the strong deflection expansion. The weak deflection limit needed no such transformation since larger impact parameters correlate to faster convergence, so a regular Taylor series expansion was used.

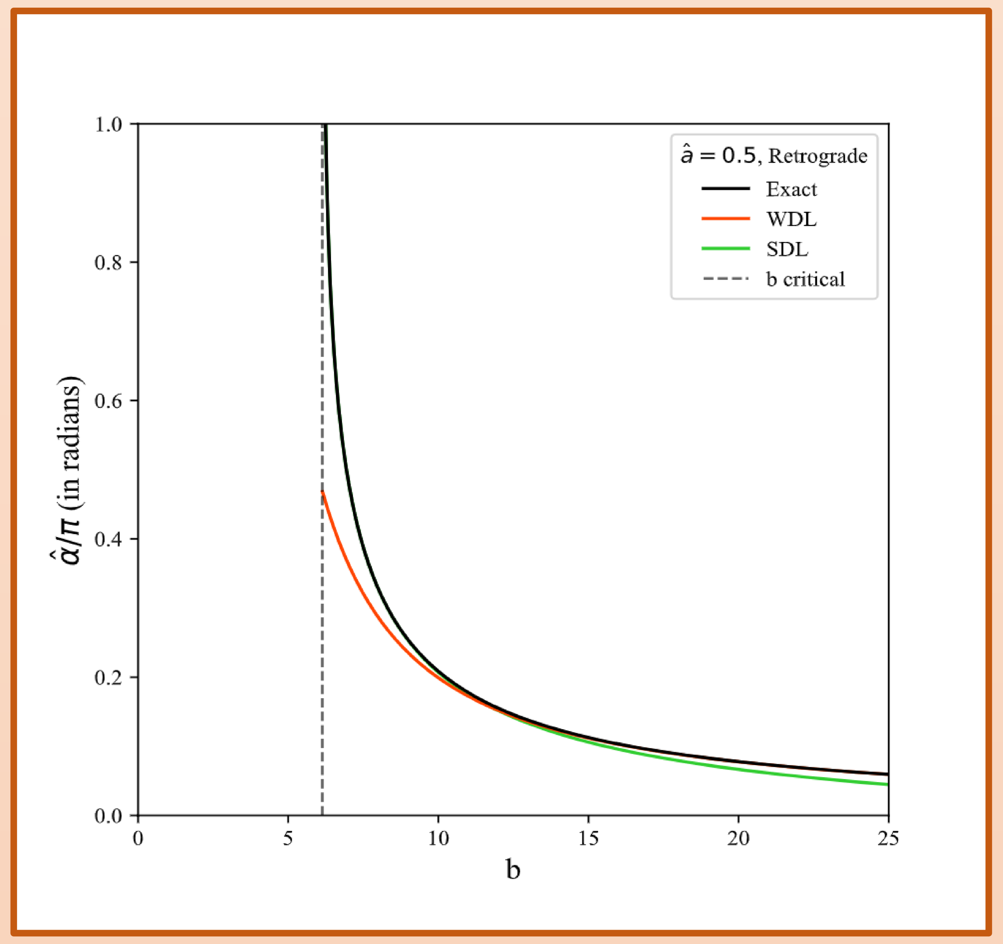


Figure 1: The 2nd order strong deflection and 6th order weak deflection plotted against the exact bending angle for $\hat{a} = 0.5$ retrograde Kerr orbits

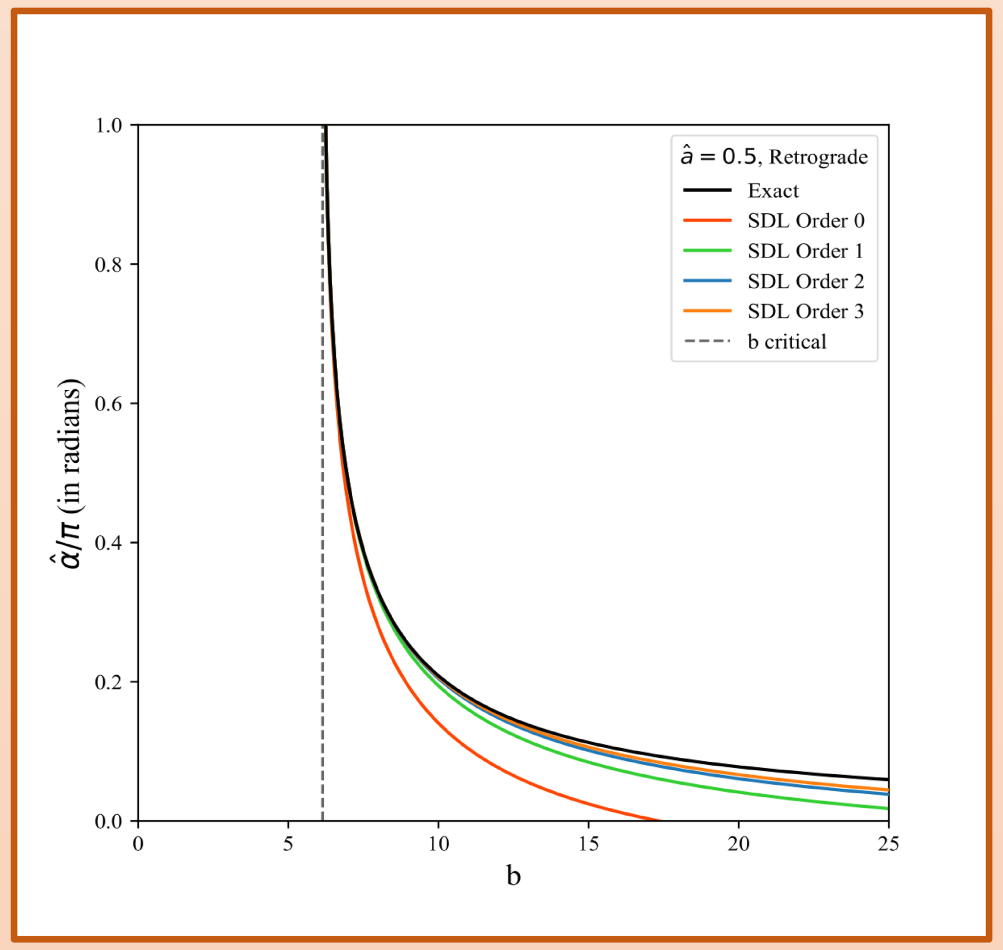
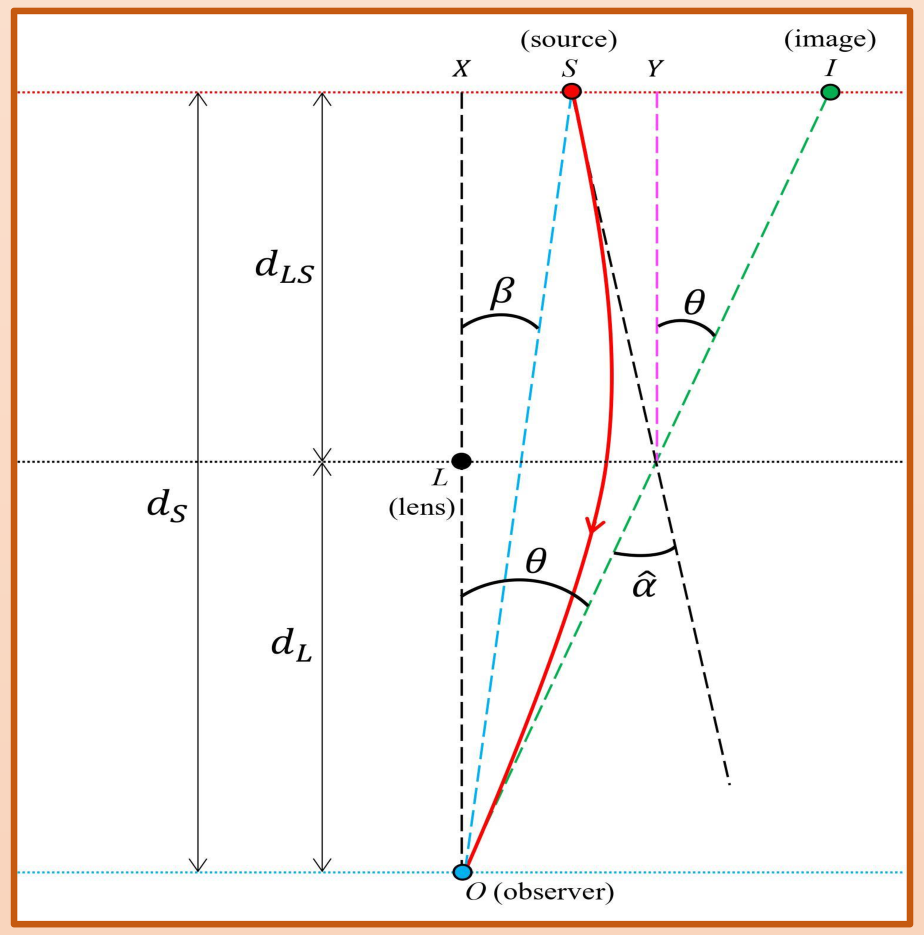


Figure 2: The progression of the strong deflection towards the exact bending angle as the order of the expansion is increased for $\hat{a} = 0.5$ retrograde Kerr orbits



$$\tan\beta = \tan\theta - \frac{d_{LS}}{d_S}\Big[\tan\theta + \tan(\alpha - \theta)\Big]$$

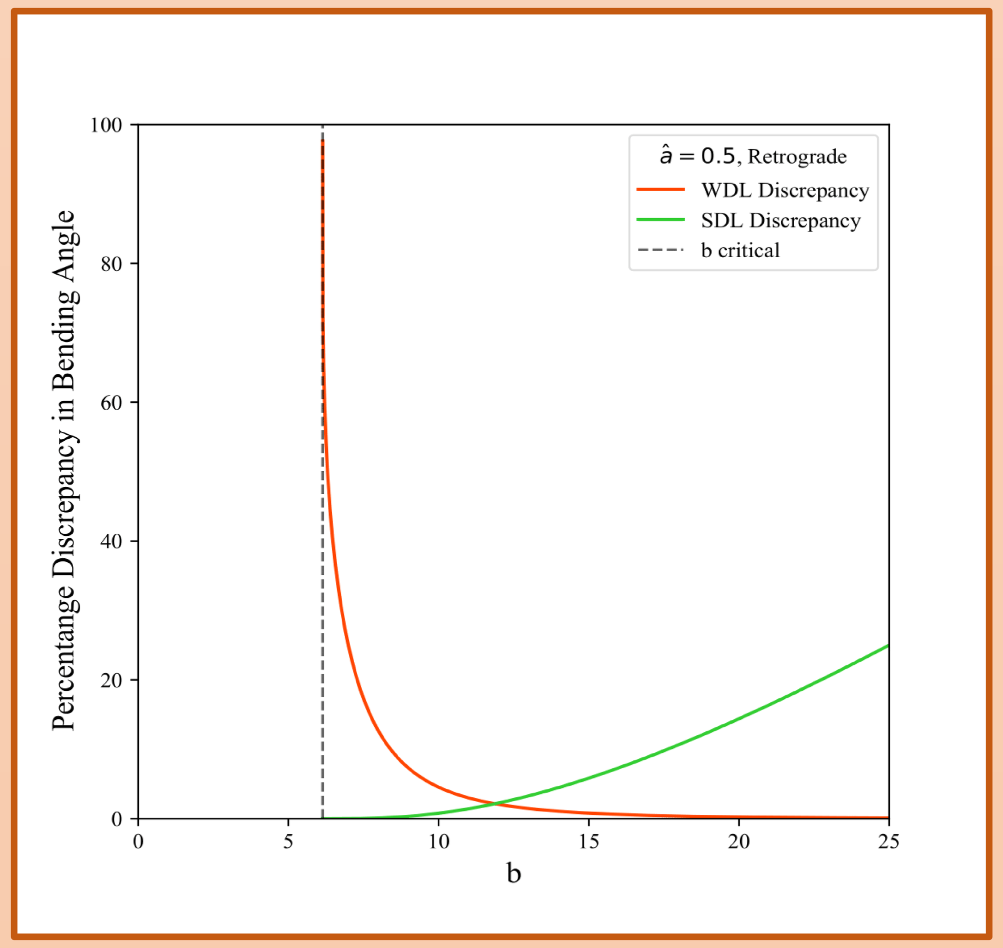


Figure 3: The discrepancies of the strong and weak deflections for retrograde Kerr orbits, from the $\hat{a} = 0.5$ spin critical impact parameter to 25M

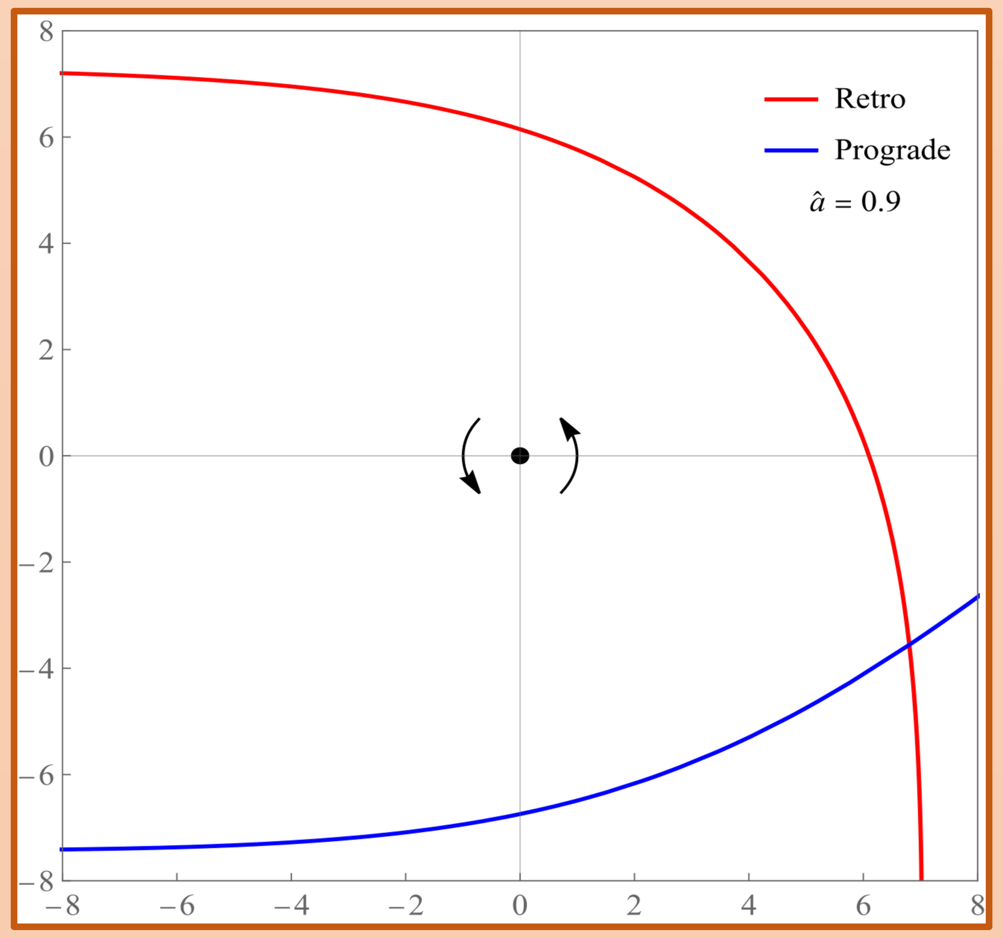


Figure 4: An observation of the asymmetry between retrograde and prograde orbits with the same impact parameter for $\hat{a} = 0.9$

$\hat{a}$	A_0	A_1	A_2	B_0	B_1
Retrograde orbits:					
0.9	0.78458	0.28022	0.16266	−0.36829	0.16359
0.8	0.80052	0.27980	0.16230	−0.36917	0.16501
0.7	0.81777	0.27921	0.16193	−0.37039	0.16679
0.6	0.83653	0.27904	0.16152	−0.37203	0.16901
0.5	0.85705	0.27870	0.16108	−0.37421	0.17181
0.4	0.87963	0.27841	0.16060	−0.37706	0.17532
0.3	0.90464	0.27815	0.16008	−0.38077	0.17974
0.2	0.93258	0.27796	0.15951	−0.38561	0.18537
0.1	0.96408	0.27783	0.15888	−0.39192	0.19258
Schwarzschild:					
0.0	1.00000	0.27778	0.15818	−0.40023	0.20195
Prograde orbits:					
0.1	1.04152	0.27783	0.15739	−0.41128	0.21430
0.2	1.09030	0.27802	0.15649	−0.42620	0.23095
0.3	1.14883	0.27838	0.15545	−0.44681	0.25399
0.4	1.22095	0.27897	0.15422	−0.47614	0.28705
0.5	1.31308	0.27987	0.15273	−0.51970	0.33687
0.6	1.43692	0.28124	0.15052	−0.58846	0.41745
0.7	1.61687	0.28332	0.14836	−0.70781	0.56282
0.8	1.91590	0.28667	0.14495	−0.95200	0.87995
0.9	2.58352	0.29273	0.11832	−1.66974	1.93623

Table 1: *Affine series coefficients for different spin parameter values.*

Results and Discussion

We were able to compute the weak deflection series analytically, given by Equation 2. It is beneficial to be able to calculate the bending angle for any value of spin and mass. However, the strong deflection series expansion was not as easily computed. Even with the transformation done on the impact parameter, the computational complexity was too high to calculate the desired series expansion in a reasonable amount of time. As such, a numerical approach was taken in order to compute expressions for various spin values, prograde and retrograde. These expressions are given by Table 1 in terms of the coefficients presented in Equation 3. Figure 2 displays the convergence of the $\hat{a} = 0.5$ expression as the order is increased, as expected from series expansions, and this behavior can be demonstrated for the other expressions as well. Interestingly, almost all coefficients in the table display a monotonic sequence. This quantitative observation has an accompanying qualitative observation: the frame-dragging effect is stronger on the retrograde side than the prograde side. That is, light orbiting in the opposite direction as the black hole is spinning will experience greater deflection than light orbiting in the same direction, as seen in Figure 4.

With expressions for both weak and strong deflection limits, the two can be combined in order to estimate the bending angle for all impact parameters. Figure 1 shows the weak and strong deflection expressions plotted against the exact bending angle for $\hat{a} = 0.5$; visually, it is easy to conclude that the two expressions do a great job of estimating the bending angle with little error. Where the strong deflection starts to divert from the exact bending angle, the weak deflection picks up at the same point. The discrepancies between the exact bending angle and the strong and weak deflection expressions are plotted in Figure 3. The discrepancy is less than five percent for all impact parameters, which makes these expressions a viable option for practical uses despite having no exact analytical expression.

Bibliography

S. V. Iyer and E. C. Hansen, Phys. Rev. D **80** 124023 (2009)

Acknowledgements

S. V. I. thanks Arlie O. Petters, C. R. Keeton, and A. B. Congdon for numerous helpful discussions. R. C. S. was supported through a NASA Cooperative Agreement awarded to the New York Space Grant Consortium. S. V. I. gratefully acknowledges the following students: E. C. Hansen, M. Colson, P. L. Kaulfuss, B. McClung, and C. E. Axelsen.